

Practice Final C1

Name: Solutions

Math 115

Instructions:

1. Do not open this exam until you are told to do so.
2. Turn off cell phones and remove headphones.
3. No calculators or electronic devices may be used during the exam.
4. Remove all resources from your desk except for your one-page "cheat sheet".
5. Write your name in at the top of each page of the exam.
6. Show your work and include relevant explanations for each problem. Unless otherwise stated, a correct answer that is not justified may receive little or no credit.
7. If you use the back of a page to show your work, clearly indicate where to look in the space allocated for the problem.

Integral Approximation Error Bounds.

If $A = \int_a^b f(x) dx$ and $|f'(x)| \leq J$ for $a \leq x \leq b$, then

$$|A - R_n| \leq \frac{J(b-a)^2}{2n} \quad |A - L_n| \leq \frac{J(b-a)^2}{2n}$$

If $A = \int_a^b f(x) dx$ and $|f''(x)| \leq K$ for $a \leq x \leq b$, then

$$|A - T_n| \leq \frac{K(b-a)^3}{12n^2} \quad |A - M_n| \leq \frac{K(b-a)^3}{24n^2}$$

Taylor's Inequality.*

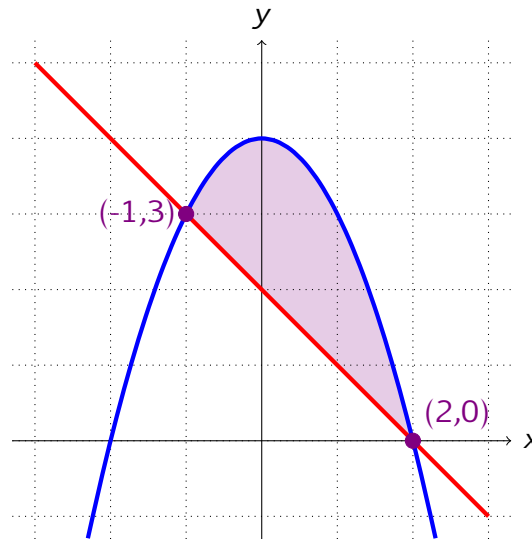
If $T_n(x)$ is the n th degree Taylor polynomial of f at a and $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad \text{for } |x - a| \leq d$$

*Taylor's Inequality is true for any interval containing the center $x = a$.

1. Let D be the region bounded by the curves $y = -x^2 + 4$ and $y = 2 - x$.

(a) Sketch the region D . Label the points where the curves intersect.



(10 points)

(b) Set up **but do not evaluate** an integral that represents the volume of the solid obtained by rotating D about the x -axis.

Solution: We can compute the volume using the washer technique.

$$\pi \int_{-1}^2 \left((-x^2 + 4)^2 - (2 - x)^2 \right) dx$$

(15 points)

(c) Set up **but do not evaluate** an integral that represents the volume of the solid obtained by rotating D about the line $x = -1$.

Solution: We can compute the volume using the shell technique.

$$2\pi \int_{-1}^2 \left((-x^2 + 4) - (2 - x) \right) (x + 1) dx$$

(15 points)

2. (a) Solve the indefinite integral, $\int \ln(x) dx$. Show your work.

Solution: Use integration by parts with $u = \ln(x)$ and $dv = dx$. Then $du = \frac{1}{x} dx$ and $v = x$.

$$\begin{aligned}\int \ln(x) dx &= x \ln(x) - \int \frac{x}{x} dx \\ &= x \ln(x) - \int 1 dx \\ &= x \ln(x) - x + C\end{aligned}$$

(10 points)

- (b) Use integration by parts to verify that for all $n > 0$,

$$\int (\ln(x))^n dx = x(\ln(x))^n - n \int (\ln(x))^{n-1} dx$$

Solution: Let $u = (\ln(x))^n$ and $dv = dx$. Then $du = \frac{n(\ln(x))^{n-1}}{x} dx$ and $v = x$.

$$\begin{aligned}\int (\ln(x))^n dx &= x(\ln(x))^n - \int \frac{x \cdot n(\ln(x))^{n-1}}{x} dx \\ &= x(\ln(x))^n - n \int (\ln(x))^{n-1} dx\end{aligned}$$

(10 points)

- (c) Use Part (b) to compute the definite integral, $\int_1^e (\ln(x))^2 dx$.

Solution: Using Part (b) with $n = 2$, we can rewrite the integral:

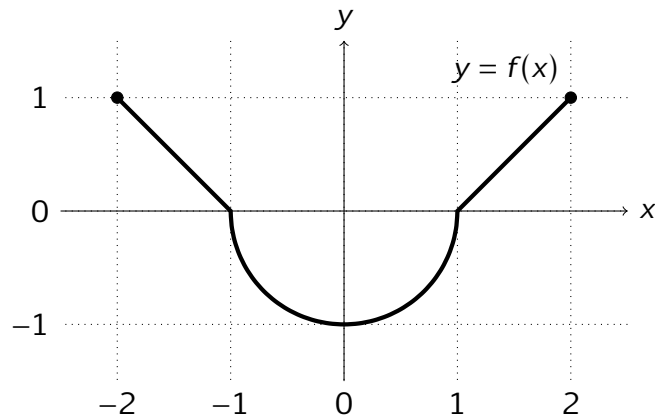
$$\int_1^e (\ln(x))^2 dx = x(\ln(x))^2 \Big|_1^e - 2 \int_1^e \ln(x) dx$$

We can rewrite the remaining integral using Part (a) to get

$$\begin{aligned}\int_1^e (\ln(x))^2 dx &= x(\ln(x))^2 \Big|_1^e - 2(x \ln(x) - x) \Big|_1^e \\ &= (e(\ln(e))^2 - (\ln(1))^2) - 2((e \ln(e) - e) - (\ln(1) - 1)) \\ &= e - 2\end{aligned}$$

(15 points)

3. Consider the function $f(x)$ whose graph is in the figure below.



(a) Compute both L_4 and R_4 , the left- and right-endpoint approximation for $\int_{-2}^2 f(x) dx$.

Solution:

$$L_4 = 1 + 0 - 1 + 0 = 0$$

$$R_4 = 0 + -1 + 0 + 1 = 0$$

(10 points)

(b) Compute T_4 , the trapezoidal approximation for $\int_{-2}^2 f(x) dx$.

Solution:

$$T_4 = \frac{1}{2} (1 + 2(0) + 2(-1) + 2(0) + 1) = 0$$

(10 points)

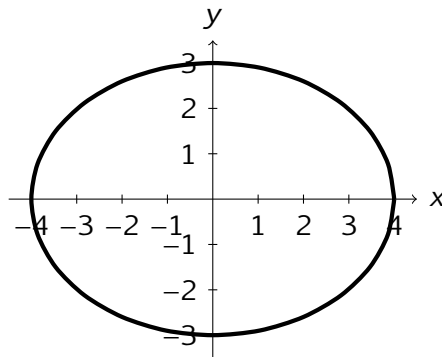
(c) Is the statement below TRUE or FALSE ? (circle one)

$$\int_{-2}^2 f(x) dx = 0$$

(5 points)

4. The ellipse in the figure below is described by the parametric equations,

$$x = 4 \sin(2t), \quad y = 3 \cos(2t)$$



(a) As t increases, the point described by the parametric equations rotates (circle one)

CLOCKWISE



COUNTERCLOCKWISE



(5 points)

(b) Find the equation of the line tangent to the curve at the point when $t = \pi/6$.

Solution: First we compute

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{-6 \sin(2t)}{8 \cos(2t)} \end{aligned}$$

So when $t = \pi/6$, the slope of the tangent line is

$$\frac{-3 \sin(\pi/3)}{4 \cos(\pi/3)} = \frac{-3(\sqrt{3}/2)}{4(1/2)} = \frac{-3\sqrt{3}}{4}$$

Also, the tangent line goes through the point:

$$x = 4 \sin(\pi/3) = 2\sqrt{3}, \quad y = 3 \cos(\pi/3) = 3/2$$

So the equation of the line tangent to the curve when $t = \pi/6$ is

$$y = \frac{3}{2} - \frac{3\sqrt{3}}{4}(x - 2\sqrt{3})$$

(15 points)

- (c) Set up **but do not evaluate** an expression representing the circumference of this ellipse.

Solution: The curve completes one full rotation from $t = 0$ to $t = \pi$. The arc length formula gives the expression

$$\int_0^{\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^{\pi} \sqrt{64\cos^2(2t) + 36\sin^2(2t)} dt$$

5. Consider the table below describing a differentiable function $f(x)$ and its derivatives.

x	0	1	2
$f(x)$	-1	2	1
$f'(x)$	1	-1	1
$f''(x)$	2	-3	-4
$f'''(x)$	-1	-1	-2

(a) Compute $\int_0^{\pi/2} \cos(x)f'(\sin(x)) dx$.

Solution: If we let $t = \sin(x)$, so $dt = \cos(x) dx$, then we can rewrite the integral with a substitution.

$$\int_0^{\pi/2} \cos(x)f'(\sin(x)) dx = \int_0^1 f'(t) dt$$

Then we can use the Fundamental Theorem of Calculus (part 2).

$$\int_0^1 f'(t) dt = f(1) - f(0) = 2 - (-1) = 3$$

(15 points)

(b) If $g(x) = \int_0^{f''(x)} t dt$, then compute $g'(1)$.

Solution: When finding $g'(x)$, we use the FTC with the Chain Rule and get

$$g'(x) = f''(x)f'''(x)$$

Now we can evaluate $g'(1) = f''(1)f'''(1) = (-3)(-1) = 3$.

(15 points)

In the parts below, we are interested in approximating $f(3)$. We are still considering the differentiable function $f(x)$ with the associated table.

x	0	1	2
$f(x)$	-1	2	1
$f'(x)$	1	-1	1
$f''(x)$	2	-3	-4
$f'''(x)$	-1	-1	-2

(c) If we use a third degree Taylor Polynomial to approximate $f(3)$, which choice for the center is likely to give the best accuracy? (circle one)

(i) $x = 0$

(ii) $x = 1$

(iii) $x = 2$

(5 points)

(d) Write down the third degree Taylor Polynomial with the center you chose in Part (c), and use it to approximate $f(3)$.

Solution:

$$\begin{aligned} T_3(x) &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3 \\ &= 1 + (x-2) - 2(x-2)^2 - \frac{1}{3}(x-2)^3 \end{aligned}$$

Now we can approximate $f(3) \approx T_3(3) = 1 + 1 - 2 - \frac{1}{3} = -\frac{1}{3}$.

(15 points)

(e) Suppose $|f^{(4)}(x)| \leq 2$ for all x . Find an upper bound on the error in your approximation from Part (d).

Solution: Using Taylor's Inequality with $M = 2$, we can bound the remainder function.

$$|R_3(x)| \leq \frac{2}{4!}|x-2|^4$$

So when $x = 3$, the error is at most $\frac{2}{24}|3-2|^4 = \frac{1}{12}$.

(15 points)

6. Determine whether each series converges or diverges. If the series converges, determine whether it converges absolutely or conditionally. Make sure to check the hypotheses of any convergence tests you use.

$$(a) \sum_{n=1}^{\infty} \frac{\sin(n)}{n^{5/3} + n}$$

Solution: First, we test for absolute convergence since $\sin(n)$ can be both positive and negative. Then we compare to the series $\sum 1/n^{5/3}$.

$$0 \leq \left| \frac{\sin(n)}{n^{5/3} + n} \right| \leq \frac{1}{n^{5/3}}$$

Since $\sum 1/n^{5/3}$ converges by the p-test ($p > 1$), our series converges absolutely by the Comparison Test.

(15 points)

$$(b) \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{3^{2n}(2n)!}$$

Solution: Since the series involves factorial, we will use the Ratio Test.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} \pi^{2(n+1)}}{3^{2(n+1)}(2(n+1))!} \cdot \frac{3^{2n}(2n)!}{(-1)^n \pi^{2n}} \right| &= \lim_{n \rightarrow \infty} \left| (-1) \cdot \frac{\pi^{2n+2}}{\pi^{2n}} \cdot \frac{3^{2n}(2n)!}{3^{2n+2}(2n+2)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{\pi^2}{3^2(2n+2)(2n+1)} \right| \\ &= \frac{\pi^2}{9} \lim_{n \rightarrow \infty} \left| \frac{1}{4n^2 + 6n + 2} \right| \\ &= 0 < 1 \end{aligned}$$

Therefore, the series converges absolutely by the Ratio Test.

(15 points)

7. Consider the series, $\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^3}$, and let $f(x) = \frac{1}{x(\ln(x))^3}$.

(a) Which of the following do we NOT need to check as a hypothesis of the Integral Test? (circle one)

i. $f(x)$ is continuous on $[3, \infty)$.

iii. $\lim_{x \rightarrow \infty} f(x) = 0$.

ii. $f(x)$ is decreasing.

iv. $f(x) \geq 0$.

(5 points)

(b) Use the Integral Test to determine whether or not the series converges. **You do not need to verify the hypotheses of the Integral Test.**

Solution: Consider the improper integral,

$$\int_3^{\infty} \frac{1}{x(\ln(x))^3} dx = \lim_{b \rightarrow \infty} \int_3^b \frac{1}{x(\ln(x))^3} dx$$

$$\text{Let } u = \ln(x), \quad du = \frac{1}{x} dx$$

$$\begin{aligned} &= \lim_{b \rightarrow \infty} \int_{\ln(3)}^{\ln(b)} \frac{1}{u^3} du \\ &= \lim_{b \rightarrow \infty} -\frac{1}{2u^2} \Big|_{\ln(3)}^{\ln(b)} \\ &= \lim_{b \rightarrow \infty} -\frac{1}{2(\ln(b))^2} + \frac{1}{2(\ln(3))^2} \\ &= \frac{1}{2(\ln(3))^2} \end{aligned}$$

Since the integral converges, our series also converges by the Integral Test.

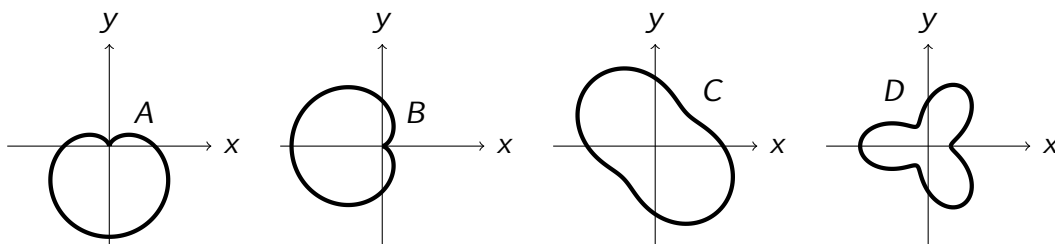
(15 points)

(c) Is the statement below TRUE or FALSE ? (circle one)

$$\text{By the Integral Test, } \sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^3} = \int_3^{\infty} f(x) dx.$$

(5 points)

8. (a) Match each of the curves with one of the polar equations below.



(i) $r = 2 - \cos(3\theta)$ matches with curve D.

(ii) $r = 2 - 2\cos(\theta)$ matches with curve B.

(iii) $r = 3 - \sin(2\theta)$ matches with curve C.

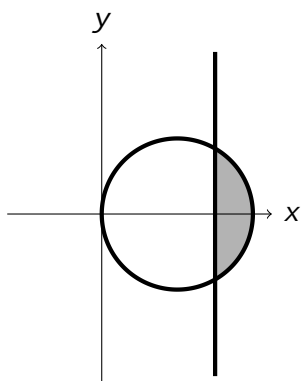
(iv) $r = 2 - 2\sin(\theta)$ matches with curve A.

(10 points)

- (b) The shaded region below is bounded by the polar curves

$$r = 4\cos(\theta) \quad \text{and} \quad r = \frac{3}{\cos(\theta)}$$

Set up **but do not evaluate** a definite integral equal to the area of this region.



Solution: First we find the points of intersection by setting the two functions equal to each other.

$$\begin{aligned} 4\cos(\theta) &= \frac{3}{\cos(\theta)} & 0 \leq \theta \leq \pi \\ \Rightarrow \cos^2(\theta) &= \frac{3}{4} \\ \Rightarrow \cos(\theta) &= \frac{\sqrt{3}}{2} \\ \Rightarrow \theta &= \pm \frac{\pi}{6} \end{aligned}$$

Now we integrate in polar coordinates to get the area between the curves.

$$\frac{1}{2} \int_{-\pi/6}^{\pi/6} \left(16\cos^2(\theta) - \frac{9}{\cos^2(\theta)} \right) d\theta$$

(15 points)

9. For this problem, you will be using the Maclaurin series, $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

(a) Find the Maclaurin series representing $x^2 \cos(x^2)$.

Solution:

$$x^2 \cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n)!}$$

The interval of convergence for this power series is $(-\infty, \infty)$.

(10 points)

(b) Use your answer to Part (a) to find a series equivalent to the definite integral,

$$\int_0^1 x^2 \cos(x^2) dx$$

Solution:

$$\begin{aligned} \int_0^1 x^2 \cos(x^2) dx &= \int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n)!} \right) dx \\ &= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(2n)!} \int_0^1 x^{4n+2} dx \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{x^{4n+3}}{4n+3} \Big|_0^1 \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{4n+3} \right) \end{aligned}$$

(15 points)

(c) Find an approximation for $\int_0^1 x^2 \cos(x^2) dx$ that is within $\frac{1}{100}$ of the actual value. Justify your answer.

Solution: First, notice that the series in Part (b) is alternating. Therefore, we can bound the error in a partial sum approximation by the magnitude of the next term.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{4n+3} \right) \approx \frac{1}{3} - \frac{1}{14}$$

The next term in the series (when $n = 2$) is $\frac{1}{4!(11)} = \frac{1}{264} < \frac{1}{100}$, so our approximation above is within 0.01 of the actual value of the integral.

(15 points)

10. For each part below, find an example that satisfies each of the following properties. Briefly justify your answers.

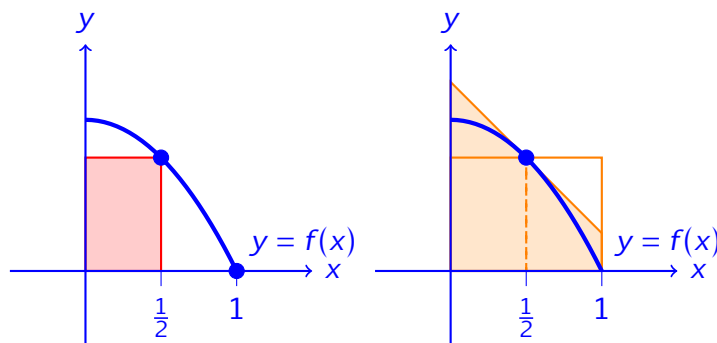
(a) Give an example of a sequence $\{b_n\}$ such that $\sum_{n=1}^{\infty} (-1)^n b_n$ diverges.

Solution: Let $b_n = (-1)^n$. Then $\sum_{n=1}^{\infty} (-1)^{2n} = \sum_{n=1}^{\infty} 1$ diverges by the Divergence Test since $\lim_{n \rightarrow \infty} 1 \neq 0$.

(b) Sketch and label the graph of a function $f(x)$ such that $R_2 < \int_0^1 f(x) dx < M_1$. (10 points)

Recall: R_n and M_n are right-endpoint and midpoint approximations for the integral.

Solution:



Since the function is decreasing, R_2 will be an underestimate, and since it is concave down, M_1 will be an overestimate.

(c) Find a function $f(x)$ for which

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{2i}{n}\right) \cdot \frac{2}{n} = e^2 - e^0$$

Solution: If $f(x) = e^x$, then by the definition of the integral,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{2i}{n}\right) \cdot \frac{2}{n} = \int_0^2 e^x dx = e^2 - e^0$$

Notice $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$.